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that it is not even desirable to do so if it were allowable. For there are serious limitations upon the applicability of Taylor's (infinite) series, whereas *any continuous function* may be treated by Weierstrass's theorem. Doubtless most scientific phenomena are continuous in the range of experimentation; they certainly are so to within the errors of observation.

It follows that the frequent occurence in science of polynomial approximations of the type (4) constitutes an illustration of Weierstrass's theorem rather than of Taylor's theorem. Quite contrary to the spirit of apologetic hesitation that is sometimes manifested by experimenters in using what are popularly known as " the first terms of a Taylor series " to represent an observed quantity, Weierstrass's theorem constitutes a complete justification for the use of formulas of the type (4) in any scientific experiment.

ON THE IMPOSSIBILITY OF CERTAIN DIOPHANTINE EQUATIONS AND SYSTEMS OF EQUATIONS.*

By R. D. CARMICHAEL, Indiana University.

INTRODUCTION.

The purpose of this paper is to give simple proofs of several theorems concerning Diophantine equations of specified form and to derive certain interesting consequences of them. All of these proofs are supposed to be novel, in whole or in part; and the results in VI, VII and VIII are believed to be new.

In §1 it is shown that the Diophantine system $q^2 + n^2 = m^2$, $m^2 + n^2 = p^2$ is without a solution (in positive integers). Several of the remaining theorems of the paper are proved by means of this one. In §2 it is shown that no numerical right triangle has a square area. In §3 the impossibility (in positive integers) of the equation $p^4 - q^4 = \alpha^2$ is proved, and several consequences of this fact are obtained. In §4 I show that the equations $m^4 - 4n^4 = \pm t^2$ are both impossible (in positive integers), and derive various consequences.

Incidentally, several proofs are given of the impossibility of the Fermat equation $x^4 + y^4 = z^4$.

The principal arguments in §§1 and 4 are made by means of Fermat's famous method of " infinite descent."

§1. *Impossibility of the simultaneous equations* $q^2 + n^2 = m^2$, $m^2 + n^2 = p^2$.

In this section we shall prove the following theorem:

THEOREM I. *There do not exist integers m, n, p, q, all different from zero, such that*

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* Presented to the American Mathematical Society, March 21, 1913.

$$q^2 + n^2 = m^2, \quad m^2 + n^2 = p^2. \tag{1}$$

It is obvious that an equivalent theorem is the following:

THEOREM II. *There do not exist integers m, n, p, q, all different from zero, such that*

$$p^2 + q^2 = 2m^2, \quad p^2 - q^2 = 2n^2. \tag{2}$$

The method of proof is to assume the existence of integers satisfying equations (1) and (2) and to show that we are thus led to a contradiction. The argument we give is an illustration of Fermat's famous method of "infinite descent."

If any two of the numbers p, q, m, n have a common prime factor t, it follows at once from (1) and (2) that all four of them have this factor. For, consider an equation in (1) or in (2) in which these two numbers occur; this equation contains a third number, and it is readily seen that this third number is divisible by t. Then from one of the equations containing the fourth number it follows that this fourth number is divisible by t. Now, let us divide each equation of system (2) by t^2; the resulting system is of the same form as (2). If any two numbers in this resulting system have a common prime factor t_1, we may divide through by t_1^2; and so on. Hence if a pair of simultaneous equations (2) exists then there exists a pair of equations of the same form in which no two of the numbers m, n, p, q have a common factor other than unity. Let this system of equations be

$$p_1^2 + q_1^2 = 2m_1^2, \quad p_1^2 - q_1^2 = 2n_1^2. \tag{3}$$

We shall show that the assumption of a pair of equations (3) leads to a contradiction. Obviously, we may without loss of generality take p_1, q_1, m_1, n_1 to be all positive; and this we do.

From the first equation in (3) it follows that p_1 and q_1 are both even or both odd; and, since they are relatively prime, it follows that they are both odd. Evidently, $p_1 > q_1$. Then we may write

$$p_1 = q_1 + 2\alpha,$$

where α is a positive integer. If we substitute this value of p_1 in the first equation of (3), the result may readily be put in the form

$$(q_1 + \alpha)^2 + \alpha^2 = m_1^2. \tag{4}$$

Since q_1 and m_1 have no common prime factor it is easy to see from this equation that α is prime to both q_1 and m_1, and hence that no two of the numbers $q_1 + \alpha$, α, m_1 have a common factor.

Now it is well known that if a, b, c are positive integers no two of which have a common prime factor, while

$$a^2 + b^2 = c^2,$$

then there exist relatively prime integers r and s, $r > s$, such that

$$c = r^2 + s^2, \quad a = 2rs, \quad b = r^2 - s^2$$

or

$$c = r^2 + s^2, \quad a = r^2 - s^2, \quad b = 2rs.$$

Hence from (4) we see that we may write

$$q_1 + \alpha = 2rs, \quad \alpha = r^2 - s^2 \tag{5}$$

or

$$q_1 + \alpha = r^2 - s^2, \quad \alpha = 2rs. \tag{6}$$

In either case we have

$$p_1^2 - q_1^2 = (p_1 - q_1)(p_1 + q_1) = 2\alpha \cdot 2(q_1 + \alpha) = 8rs(r^2 - s^2).$$

If we substitute in the second equation of (3) and divide by 2 we have

$$4rs(r^2 - s^2) = n_1^2.$$

From this equation and the fact that r and s are relatively prime it follows at once that r, s, $r^2 - s^2$ are all square numbers; say,

$$r = u^2, \quad s = v^2, \quad r^2 - s^2 = w^2.$$

Now $r - s$ and $r + s$ can have no common factor other than 1 or 2; hence from

$$w^2 = r^2 - s^2 = (r - s)(r + s) = (u^2 - v^2)(u^2 + v^2)$$

we see that either

$$u^2 + v^2 = 2w_1^2, \quad u^2 - v^2 = 2w_2^2 \tag{7}$$

or

$$u^2 + v^2 = w_1^2, \quad u^2 - v^2 = w_2^2.$$

And if it is the latter case which arises, then

$$w_1^2 + w_2^2 = 2u^2, \quad w_1^2 - w_2^2 = 2v^2. \tag{8}$$

Hence, assuming equations of the form (2) we are led either to equations (7) or to equations (8); that is, we are led to new equations of the form with which we started. Let us write the equations thus:

$$p_2^2 + q_2^2 = 2m_2^2, \quad p_2^2 - q_2^2 = 2n_2^2; \tag{9}$$

that is, system (9) is identical with that one of systems (7), (8) which actually arises.

Now from (5) and (6) and the relations $p_1 = q_1 + 2\alpha$, $r > s$, we see that

$$p_1 = 2rs + r^2 - s^2 > 2s^2 + r^2 - s^2 = r^2 + s^2 = u^4 + v^4.$$

Hence $u < p_1$. Also,

$$w_1^2 \underset{<}{=} w^2 \underset{<}{=} r + s < r^2 + s^2.$$

Hence $w_1 < p_1$. Since u and w_1 are both less than p_1 it follows that p_2 is less than p_1. Hence, obviously, $p_2 < | p |$. Moreover, it is clear that all the numbers p_2, q_2, m_2, n_2 are different from zero.

From these results we have the following conclusion: If we assume a system of the form (2) we are led to a new system (9) of the same form; and in the new system $|p_2|$ is less than $|p|$.

Now if we start with (9) and carry out a similar argument we shall be led to a new system

$$p_3^2 + q_3^2 = 2m_3^2, \quad p_3^2 - q_3^2 = 2n_3^2,$$

with the relation $|p_3| < |p_2|$; and so on *ad infinitum.* But, since there is only a finite number of positive integers less than the given integer $|p|$ this is impossible. We are thus led to a contradiction; whence we conclude at once the truth of II, and likewise of I.

§2. *The area of a numerical right triangle is never a square number.*

This famous theorem of Fermat is readily proved by means of the results in the preceding section. Let the sides and hypotenuse of a numerical right triangle be u, v and w, respectively. The area of this triangle is $\frac{1}{2}uv$. If we assume this to be a square number t^2 we should have the following simultaneous Diophantine equations

$$u^2 + v^2 = w^2, \quad uv = 2t^2. \tag{10}$$

We shall prove our theorem by showing that the assumption of such a system leads to a contradiction.

If any two of the numbers u, v, w have a common prime factor p then the remaining one also has this factor, as one sees readily from the first equation in (10). From the second equation in (10) it follows that t also has the same factor. Then if we put $u = pu_1$, $v = pv_1$, $w = pw_1$, $t = pt_1$, we have

$$u_1^2 + v_1^2 = w_1^2, \quad u_1v_1 = 2t_1^2,$$

a system of the same form as (10). It is clear that we may start with this new system and proceed in the same manner as before, and so on, until we arrive at a system

$$\bar{u}^2 + \bar{v}^2 = \bar{w}^2, \quad \bar{u}\bar{v} = 2\bar{t}^2, \tag{11}$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are prime each to each.

It is well known that the general solution of the first equation (11) may be written in one of the forms

$$\bar{u} = 2ab, \qquad \bar{v} = a^2 - b^2, \quad \bar{w} = a^2 + b^2;$$

$$\bar{u} = a^2 - b^2, \quad \bar{v} = 2ab, \qquad \bar{w} = a^2 + b^2.$$

Then from the second equation in (11) we have

$$\bar{t}^2 = ab(a^2 - b^2) = ab(a - b)(a + b).$$

It is easy to see that no two of the numbers a, b, $a - b$, $a + b$ in the last member of this equation have a common factor; for, if so, $\bar{u}$ and $\bar{v}$ would have a common factor, contrary to hypothesis. Hence each of these four numbers is a square. That is, we have equations of the form

$$a = m^2, \quad b = n^2, \quad a + b = p^2, \quad a - b = q^2;$$

whence

$$m^2 - n^2 = q^2, \quad m^2 + n^2 = p^2.$$

But, according to theorem I, no such system of equations can exist.

Hence we have the following theorem:

THEOREM III. *The area of a numerical right triangle is never a square number.*

COROLLARY. *The equation* $\alpha^4 + 4\beta^4 = \gamma^2$ *is impossible in integers* α, β, γ, *all of which are different from zero.*

For, from $\alpha^4 + 4\beta^4 = \gamma^2$ it would follow that a right triangle whose sides and hypotenuse are respectively α^2, $2\beta^2$, γ has the square area $\alpha^2\beta^2$, which is impossible.

In the next section additional proofs will be given of this corollary and theorem.

§3. *Impossibility of the equation* $p^4 - q^4 = \alpha^2$. *Consequences.*

If there exists a set of integers p, q, α all different from zero, such that

$$p^4 - q^4 = \alpha^2, \tag{12}$$

it is easy to see that there exists such a set having also the further property that they are prime each to each. For, if any two of the numbers contain the prime factor t, the third contains this factor also and the equation may be divided through by t^4. Continuing this process one arrives finally at an equation of the form (12) in which p, q, α are prime each to each.

We shall now show that the assumption of such an equation (12) leads to a contradiction of theorem I or of theorem II. We have

$$(p^2 - q^2)(p^2 + q^2) = \alpha^2.$$

Since p and q are relatively prime the two factors of the first member of this equation evidently have the greatest common factor 1 or 2; for the greatest common factor is a divisor of $2q^2$, the difference of the numbers. Hence $p^2 - q^2$ and $p^2 + q^2$ are either both square numbers or both the double of square numbers. That is, we have a system of the form

$$p^2 - q^2 = m^2, \quad p^2 + q^2 = n^2,$$

or a system of the form

$$p^2 - q^2 = 2m^2, \quad p^2 + q^2 = 2n^2.$$

But, according to theorems I and II, each of these systems is impossible.

Thus we have the following theorem:

THEOREM IV. *There do not exist integers* p, q, α, *all different from zero, such that*

$$p^4 - q^4 = \alpha^2;$$

that is, the difference of two fourth power integers cannot be a square.

Second proof. This theorem may also be readily proved by means of theorem III. For, assuming (12) we have

$$(p^4 - q^4)p^2q^2 = p^2q^2\alpha^2. \tag{13}$$

But, obviously,

$$(2p^2q^2)^2 + (p^4 - q^4)^2 = (p^4 + q^4)^2. \tag{14}$$

Now, from (13) we see that the numerical right triangle determined by (14) has its area $p^2q^2(p^4 - q^4)$ equal to the square number $p^2q^2\alpha^2$. But this is impossible. Hence no equation of the form (12) exists.

COROLLARY. *The system* $p^2 - q^2 = km^2$, $p^2 + q^2 = kn^2$ *is impossible in integers* p, q, k, m, n, *all of which are different from zero.*

For, if we take the product of these two equations member by member we have as a result an equation which by theorem IV is impossible.

Since from $x^4 + y^4 = z^4$, we have $z^4 - x^4 = (y^2)^2$ the above theorem leads to the following:

THEOREM V. *The equation* $x^4 + y^4 = z^4$ *has no solution in integers all of which are different from zero.*

By means of theorem IV one may readily prove the corollary to theorem III. Let us assume an equation of the form

$$\alpha^4 + 4\beta^4 = \gamma^2.$$

If α and β have a common prime factor p, γ has also this factor, and we may divide the equation by p^4, the resulting equation being of the same form as the original one. This process may be repeated until we have an equation

$$\alpha_1^4 + 4\beta_1^4 = \gamma_1^2,$$

where α_1 and β_1 are relatively prime. It is then easy to see that γ_1 has no odd factor in common with either α_1 or β_1. Suppose that α_1 and γ_1 have the factor 2 in common. We may then divide the equation by 4, thus obtaining a new equation of the same form. We may continue this process until the α and the γ of the resulting equation are not both even. It is then obvious that both must be odd. Let this equation be

$$\bar{\alpha}^4 + 4\bar{\beta}^4 = \bar{\gamma}^2.$$

Then $\bar{\alpha}^2$, $2\bar{\beta}^2$ and $\bar{\gamma}$ are relatively prime. Then from the theory of numerical right triangles it follows that we have equations of the form

$$\bar{\beta}^2 = ab, \qquad \bar{\alpha}^2 = a^2 - b^2, \qquad \bar{\gamma} = a^2 + b^2;$$

where a and b are relatively prime positive integers. From the first of these equations we see that a and b are both square numbers. Then the second of the equations is of the form (12). Since no equation of this form exists we conclude the impossibility of the equation $\alpha^4 + 4\beta^4 = \gamma^2$; that is, we have the corollary of theorem III.

Having demonstrated this corollary without the use of theorem III we may now employ the corollary to prove III itself. Let the argument read as in the preceding case down to equations (11). From the second equation in (11) we see that $\bar{u}$ and $\bar{v}$ have this property: One of them is a square number, say α^2, and the other is twice a square number, say $2\beta^2$. Then the first equation in (11) takes the form $\alpha^4 + 4\beta^4 = \bar{w}^2$. This equation being impossible we conclude the truth of theorem III.

§4. *Impossibility of the equations* $m^4 - 4n^4 = \pm t^2$. *Consequences.*

We shall assume the existence of one of the equations

$$m^4 - 4n^4 = \pm t^2, \tag{15}$$

in which m, n, t are all different from zero, and show that we are led to a contradiction. If any two of the numbers m, n, t have a common odd prime factor p then all three of them have this factor, and the equation may be divided through by p^4. The new equation thus obtained is of the same form as the original one. This process may be repeated until an equation

$$m_1^4 - 4n_1^4 = \pm t_1^2$$

is obtained in which no two of the numbers m_1, n_1, t_1 have a common odd prime factor.

If t_1 is even it is obvious that m_1 is also even, and therefore the above equation may be divided by 4; a result of the form

$$n_1^4 - 4m_2^4 = \mp t_2^2$$

is obtained. The process can be continued until an equation

$$m_3^4 - 4n_3^4 = \pm t_3^2$$

is obtained in which t_3 is an odd number. Then m_3 is also odd. If the second member of the above equation has the minus sign then we have

$$(m_3^2)^2 + t_3^2 = 4n_3^4;$$

that is, we have the sum of two odd squares divisible by 4, which is impossible since an odd square is congruent to 1 modulo 4. Hence we must have

$$4n_3^4 + t_3^2 = m_3^4. \tag{16}$$

Moreover, it is obvious that no two of the numbers n_3, t_3, m_3 have a common prime factor and that all of them are different from zero.

From (16) and the theory of numerical right triangles we have equations of the following form:

$$n_3^2 = rs, \qquad t_3 = r^2 - s^2, \qquad m_3^2 = r^2 + s^2,$$

where r and s are relatively prime positive integers. From the first of these equations it follows that r and s are squares; say, $r = \rho^2$, $s = \sigma^2$. Then from the last equation we have

$$\rho^4 + \sigma^4 = m_3^2.$$

It is easy to see that ρ, σ and m_3 are prime each to each.

For this equation we have the general solution

$$m_3 = r_1^2 + s_1^2, \quad \rho^2 = 2r_1s_1, \quad \sigma^2 = r_1^2 - s_1^2,$$

or the general solution

$$m_3 = r_1^2 + s_1^2, \qquad \rho^2 = r_1^2 - s_1^2, \qquad \sigma^2 = 2r_1s_1.$$

In either case we see that $2r_1s_1$ and $r_1^2 - s_1^2$ are both squares, while r_1 and s_1 are relatively prime and one of them is even. From the relation $r_1^2 - s_1^2 =$ square, it follows that r_1 is odd, since otherwise we should have the sum of two odd squares equal to the even square r_1^2, which is impossible. Hence s_1 is even. But $2r_1s_1 =$ square. Then, since r_1 is odd we have r_1 equal to a square. Then $2s_1$ is a square, or s_1 is twice a square. Then we may write $r_1 = \rho_1^2$, $s_1 = 2\sigma_1^2$. Hence we have an equation of the form

$$\rho_1^4 - 4\sigma_1^4 = w_1^2$$

since $r_1^2 - s_1^2$ is a square; that is, we have

$$4\sigma_1^4 + w_1^2 = \rho_1^4. \tag{17}$$

Now the last equation has been obtained solely from (16). Moreover it is obvious that no one of the numbers σ_1, w_1, ρ_1 is zero. Also, we have

$$n_3^2 = rs = \rho^2\sigma^2 - 2r_1s_1(r_1^2 - s_1^2) = 4\rho_1^2\sigma_1^2(\rho_1^4 - 4\sigma_1^4) = 4\rho_1^2\sigma_1^2w_1^2.$$

Hence $\sigma_1 < n_3$. Similarly, starting from (17) we should be led to an equation

$$4\sigma_2^4 + w_2^2 = \rho_2^4,$$

where $\sigma_2 < \sigma_1$; and so on indefinitely. But such a recursion is impossible.

We have been led to this contradiction by the initial assumption that an equation of the form (15) is possible. Hence we have the following theorem:

THEOREM VI. *Neither of the equations* $m^4 - 4n^4 = \pm t^2$ *is possible in integers* m, n, *and* t, *all of which are different from zero.*

COROLLARY. *The system of equations* $p^2 - 2q^2 = km^2$, $p^2 + 2q^2 = \pm kn^2$ *is not possible in integers* p, q, k, m, n, *all of which are different from zero.*

For, if we take the product of these equations member by member we have as a result an equation which by theorem VI is impossible.

From the corollary to theorem VI we have also the following interesting result:

THEOREM VII. *The area of a numerical right triangle is not twice a square number.*

For, if there exists a set of numbers u, v, w such that

$$u^2 + v^2 = w^2, \qquad uv = 4k^2 = t^2$$

then it is easy to see that

$$(u + v)^2 = w^2 + 2t^2,$$

$$(u - v)^2 = w^2 - 2t^2.$$

Since from the preceding corollary these equations are impossible, the truth of the theorem follows at once.

From theorems III and VII we have the following corollary:

COROLLARY. *The number expressing the area of a numerical right triangle has at least one odd prime factor entering to an odd power.*

Now a number of the form $rs(r^2 - s^2)$, where r and s are different positive integers, is the area of the numerical right triangle determined by the equation

$$(2rs)^2 + (r^2 - s^2)^2 = (r^2 + s^2)^2.$$

Put $r = \rho^2$ and $s = \sigma^2$. Then from the above corollary we see that $\rho^2\sigma^2(\rho^4 - \sigma^4)$ has some odd prime factor entering into it to an odd power. Since every prime factor of $\rho^2\sigma^2$ obviously enters into $\rho^2\sigma^2$ to an even power, we have the following result:

THEOREM VIII. *The number $\rho^4 - \sigma^4$, in which ρ and σ are different positive integers, has always an odd prime factor entering into it to an odd power.*

As immediate corollaries of this theorem we have theorems IV and V in §3.

By means of theorem VII we may also prove the following theorem:

THEOREM IX. *The equation $m^4 + n^4 = \alpha^2$ is impossible in integers all of which are different from zero.*

For, if such an equation exists we have a right triangle $(m^2)^2 + (n^2)^2 = \alpha^2$ whose area $\frac{1}{2}m^2n^2$ is twice a square number, which is impossible.

Another proof of theorem V by means of theorem IX is immediate. Theorems IV and IX, taken together, may be stated in the form:

THEOREM X. *In a numerical right triangle $a^2 + b^2 = c^2$, not more than one of the numbers a, b, c is a square.*

PROBLEMS AND QUESTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

ALGEBRA.

392. Proposed by H. PRIME, Boston, Mass.

The floor and ceiling of a room h feet high are parallel and horizontal. In the middle of the ceiling is a vertical circular tube w feet in diameter extending upward from the ceiling. From the room the longest possible inflexible rod is put up the tube. When the rod is in contact with the opposite sides of the tube and the floor, what is the expression for the tangent of the acute angle made by the rod and the floor in terms of h and w, considering the rod as an air line. To be solved without the calculus. [From *The Maine Farmers' Almanac*, 1913.]

393. Proposed by H. E. TREFETHEN, Waterville, Maine.

Expand log $[(\sin x)/x]$ in terms of x.

394. Proposed by E. B. ESCOTT, University of Michigan.

Solve the equation $\sin^2 x \sin^2 2x = 5/16$.

395. Proposed by V. M. SPUNAR, Chicago, Ill.

Solve the system of equations $x_1^2x_2 = a_1$; $x_2^2x_3 = a_2$; $x_3^2x_4 = a_3$; $\cdots$ $x_n^2x_1 = a_n$.